A Generalized Fractional Calculus of Variations*

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Abstract

We study incommensurate fractional variational problems in terms of a generalized fractional integral with Lagrangians depending on classical derivatives and generalized fractional integrals and derivatives. We obtain necessary optimality conditions for the basic and isoperimetric problems, transversality conditions for free boundary value problems, and a generalized Noether type theorem.

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1 Introduction

Till recently, it was believed that Lagrangian and Hamiltonian mechanics were not valid in the presence of nonconservative forces such as friction [23]. In the last years, however, several approaches have been investigated in order to find a Lagrangian or a Hamiltonian description for classes of dissipative (or dissipative-looking) systems [7, 8, 22, 27, 37]. One possibility to have a Lagrangian and a Hamiltonian formulation, for both conservative and nonconservative systems, was proposed by Fred Riewe in 1996 and consists in using fractional derivatives [38, 39]. Riewe's papers [38, 39] gave rise to a new and important research field, called *the fractional calculus of variations* [24]. Nowadays the subject is of strong interest, and many results of variational analysis were extended to the non-integer case (see, e.g., [2–6, 9, 12, 26, 29, 33]). Here we study problems of calculus of variations with generalized fractional operators [1, 30, 31]. Generalized fractional integrals are given as a linear combination of left and right fractional integrals with general kernels. Generalized fractional Riemann–Liouville and Caputo derivatives are defined as a composition of classical derivatives and generalized fractional integrals. In a first problem, we ask how to determine the extremizers of a functional defined by a generalized fractional integral involving n generalized fractional Caputo derivatives and n generalized fractional integrals. All these operators have different (non-integer) orders. We obtain necessary optimality conditions, and in the case of free boundary values, also natural boundary conditions. Next, we derive Euler–Lagrange type equations for an extended isoperimetric problem and we obtain a Noether type theorem.

The text is organized as follows. In Section 2 we give the definitions and main properties of the generalized fractional operators. We prove Euler–Lagrange equations for the fundamental generalized problem in Section 3, and natural boundary conditions for free boundary value problems in

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Section 4. Section 5 is devoted to the generalized isoperimetric problem and Section 6 to Noether's theorem. Finally, in Section 7 we present an application of our results to the damped harmonic oscillator.

2 Preliminaries

We start by defining the generalized fractional operators [1]. As particular cases, by choosing appropriate kernels, such operators are reduced to the standard fractional integrals and derivatives of fractional calculus (see, e.g., [20,21,34]). Throughout the text, α denotes a real number between zero and one. Following [3], we use round brackets for the arguments of functions, and square brackets for the arguments of operators.

Definition 1 (The generalized fractional integral)**.** *The operator K_P^α is given by*

$$K_P^\alpha[f](x) := K_P^\alpha[t \mapsto f(t)](x) = p\int_a^x k_\alpha(x,t)f(t)dt + q\int_x^b k_\alpha(t,x)f(t)dt,$$

where $P = \langle a, x, b, p, q\rangle$ is the parameter set *(p-set for brevity), $x \in [a,b]$, p, q are real numbers, and $k_\alpha(x,t)$ is a kernel which may depend on α. The operator K_P^α is referred as the* operator K *(K-op for simplicity) of order α and p-set P.*

Note that if we define

$$G(x,t) := \begin{cases} pk_\alpha(x,t) & \text{if } t < x, \\ qk_\alpha(t,x) & \text{if } t \geq x, \end{cases}$$

then the operator K_P^α can be written in the form

$$K_P^\alpha[f](x) = K_P^\alpha[t \mapsto f(t)](x) = \int_a^b G(x,t)f(t)dt.$$

Thus, the generalized fractional integral is a Fredholm operator, one of the oldest and most respectable class of operators that arise in the theory of integral equations [16,35].

Example 2. *1. Let $k_\alpha(t-\tau) = \frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}$ and $0 < \alpha < 1$. If $P = \langle a, t, b, 1, 0\rangle$, then*

$$K_P^\alpha[f](t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau =: {}_aI_t^\alpha[f](t)$$

is the left Riemann–Liouville fractional integral of order α; if $P = \langle a, t, b, 1, 0\rangle$, then

$$K_P^\alpha[f](t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}f(\tau)d\tau =: {}_tI_b^\alpha[f](t)$$

is the right Riemann–Liouville fractional integral of order α.

2. For $k_\alpha(t-\tau) = \frac{1}{\Gamma(\alpha(t,\tau))}(t-\tau)^{\alpha(t,\tau)-1}$ and $P = \langle a, t, b, 1, 0\rangle$

$$K_P^\alpha[f](t) = \int_a^t \frac{1}{\Gamma(\alpha(t,\tau)}(t-\tau)^{\alpha(t,\tau)-1}f(\tau)d\tau =: {}_aI_t^{\alpha(t,\cdot)}[f](t)$$

is the left Riemann–Liouville fractional integral of variable order $\alpha(t,\tau)$, and for $P = \langle a, t, b, 0, 1\rangle$

$$K_P^\alpha[f](t) = \int_t^b \frac{1}{\Gamma(\alpha(\tau,t))}(\tau-t)^{\alpha(t,\tau)-1}f(\tau)d\tau =: {}_tI_b^{\alpha(\cdot,t)}[f](t)$$

is the right Riemann–Liouville fractional integral of variable order $\alpha(t,\tau)$ [32].

3. *For $0<\alpha<1$, $k_\alpha(t,\tau)=\frac{1}{\Gamma(\alpha)}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{1}{\tau}$ and $P=\langle a,t,b,1,0\rangle$, the operator K_P^α reduces to the left Hadamard fractional integral [36],*

$$K_P^\alpha[f](t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{f(\tau)d\tau}{\tau}=:{}_aJ_t^\alpha[f](t),$$

and for $P=\langle a,t,b,0,1\rangle$ operator K_P reduces to the right Hadamard fractional integral,

$$K_P^\alpha[f](t)=\frac{1}{\Gamma(\alpha)}\int_t^b\left(\log\frac{\tau}{t}\right)^{\alpha-1}\frac{f(\tau)d\tau}{\tau}=:{}_tJ_b^\alpha[f](t).$$

4. *Generalized fractional integrals can be also reduced to, e.g., Riesz, Katugampola or Kilbas fractional operators. Their definitions can be found in [18–20].*

Next results yield boundedness of the generalized fractional integral.

Theorem 3 (cf. Example 6 of [16])**.** *Let $\alpha\in(0,1)$ and $P=\langle a,x,b,p,q\rangle$. If k_α is a square integrable function on the square $\Delta=[a,b]\times[a,b]$, then $K_P^\alpha: L_2([a,b])\to L_2([a,b])$ is well defined, linear, and bounded operator.*

Theorem 4 (cf. [30,31])**.** *Let $k_\alpha\in L_1([0,b-a])$ be a difference kernel, that is, $k_\alpha(x,t)=k_\alpha(x-t)$. Then, $K_P^\alpha: L_1([a,b])\to L_1([a,b])$ is a well defined bounded and linear operator.*

Theorem 5 (cf. Theorem 2.4 of [31])**.** *Let $P=\langle a,x,b,p,q\rangle$. If $k_{1-\alpha}$ is a difference kernel, $k_{1-\alpha}\in L_1([0,b-a])$ and $f\in AC([a,b])$, then $K_P^{1-\alpha}[f]$ belongs to $AC([a,b])$.*

The generalized fractional derivatives A_P^α and B_P^α are defined in terms of the generalized fractional integral K-op.

Definition 6 (Generalized Riemann–Liouville fractional derivative)**.** *Let P be a given parameter set and $0<\alpha<1$. The operator A_P^α is defined by $A_P^\alpha:=D\circ K_P^{1-\alpha}$, where D denotes the standard derivative operator, and is referred as the* operator A *(A-op) of order α and p-set P.*

Remark 7. *Operator A is well-defined for all functions f such that $K_P^{1-\alpha}[f]$ is differentiable. Theorem 5 assure us that the domain of A is nonempty.*

Definition 8 (Generalized Caputo fractional derivative)**.** *Let P be a given parameter set and $\alpha\in(0,1)$. The operator B_P^α is defined by $B_P^\alpha:=K_P^{1-\alpha}\circ D$, where D denotes the standard derivative operator, and is referred as the* operator B *(B-op) of order α and p-set P.*

Remark 9. *Operator B is well-defined for differentiable functions.*

Example 10. *The standard Riemann–Liouville and Caputo fractional derivatives (see, e.g., [20, 21, 34]) are easily obtained from the general kernel operators A_P^α and B_P^α, respectively. Let $k_\alpha(t-\tau)=\frac{1}{\Gamma(1-\alpha)}(t-\tau)^{-\alpha}$, $\alpha\in(0,1)$. If $P=\langle a,t,b,1,0\rangle$, then*

$$A_P^\alpha[f](t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t(t-\tau)^{-\alpha}f(\tau)d\tau=:{}_aD_t^\alpha[f](t)$$

is the standard left Riemann–Liouville fractional derivative of order α, while

$$B_P^\alpha[f](t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t(t-\tau)^{-\alpha}f'(\tau)d\tau=:{}_a^CD_t^\alpha[f](t)$$

is the standard left Caputo fractional derivative of order α; if $P=\langle a,t,b,0,1\rangle$, then

$$-A_P^\alpha[f](t)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b(\tau-t)^{-\alpha}f(\tau)d\tau=:{}_tD_b^\alpha[f](t)$$

is the standard right Riemann–Liouville fractional derivative of order α, while

$$-B_P^\alpha[f](t) = -\frac{1}{\Gamma(1-\alpha)} \int\limits_t^b (\tau - t)^{-\alpha} f'(\tau) d\tau =: {}_t^C D_b^\alpha[f](t)$$

is the standard right Caputo fractional derivative of order α.

The following theorems give integration by parts formulas for operators A, B and K. For detailed proofs we refer the reader to [30, 31].

Theorem 11. *Let $\alpha \in (0,1)$, $P = \langle a,t,b,p,q \rangle$, k_α be a square-integrable function on $\Delta = [a,b] \times [a,b]$, and $f, g \in L_2([a,b])$. The generalized fractional integral K_P^α satisfies the integration by parts formula*

$$\int\limits_a^b g(x) K_P^\alpha[f](x) dx = \int\limits_a^b f(x) K_{P^*}^\alpha[g](x) dx, \tag{1}$$

where $P^* =< a,t,b,q,p >$.

Theorem 12. *Let $\alpha \in (0,1)$, $P = \langle a,t,b,p,q \rangle$, and k_α be a square integrable function on $\Delta = [a,b] \times [a,b]$. If functions $f, K_{P^*}^{1-\alpha}[g] \in AC([a,b])$, then*

$$\int\limits_a^b g(x) B_P^\alpha[f](x) dx = f(x) K_{P^*}^{1-\alpha}[g](x)\Big|_a^b - \int_a^b f(x) A_{P^*}^\alpha[g](x) dx, \tag{2}$$

where $P^* =< a,t,b,q,p >$.

Theorem 13. *Let $0 < \alpha < 1$, $P = \langle a,x,b,p,q \rangle$, and k_α be a difference kernel such that $k_\alpha \in L_1[0, b-a]$. If $f \in L_1([a,b])$ and $g \in C([a,b])$, then the operator K_P^α satisfies the integration by parts formula* (1).

Theorem 14. *Let $\alpha \in (0,1)$, $P = \langle a,t,b,p,q \rangle$, and $k_\alpha \in L_1([0, b-a])$ be a difference kernel. If functions $f, g \in AC([a,b])$, then formula* (2) *holds.*

For $\mathbf{f} = [f_1, \ldots, f_N] : [a,b] \to \mathbb{R}^N$, where $N \in \mathbb{N}$, we put

$$\begin{aligned} A_P^\alpha[\mathbf{f}](x) &:= [A_P^\alpha[f_1](x), \ldots, A_P^\alpha[f_N](x)], \\ B_P^\alpha[\mathbf{f}](x) &:= [B_P^\alpha[f_1](x), \ldots, B_P^\alpha[f_N](x)], \\ K_P^\alpha[\mathbf{f}](x) &:= [K_P^\alpha[f_1](x), \ldots, K_P^\alpha[f_N](x)]. \end{aligned}$$

3 The generalized fundamental variational problem

We consider the problem of finding a function $\mathbf{y} = [y_1, \ldots, y_N]$ that gives an extremum (minimum or maximum) to the functional

$$\mathcal{J}(\mathbf{y}) = K_P^\alpha\left[t \mapsto F\left(t, \mathbf{y}(t), \mathbf{y}'(t), B_{P_1}^{\beta_1}[\mathbf{y}](t), \ldots, B_{P_n}^{\beta_n}[\mathbf{y}](t), K_{R_1}^{\gamma_1}[\mathbf{y}](t), \ldots, K_{R_m}^{\gamma_m}[\mathbf{y}](t)\right)\right](b) \tag{3}$$

subject to the boundary conditions

$$\mathbf{y}(a) = \mathbf{y}_a, \quad \mathbf{y}(b) = \mathbf{y}_b, \tag{4}$$

where $\alpha, \beta_i, \gamma_k \in (0,1)$, $P =< a,b,b,1,0 >$, $P_i =< a,t,b,p_i,q_i >$, and $R_k =< a,t,b,r_k,s_k >$, $i = 1, \ldots, n$, $k = 1, \ldots, m$. For simplicity of notation, we introduce the operator $\{\cdot\}_{P_D,R_I}^{\beta,\gamma}$ defined by

$$\{\mathbf{y}\}_{P_D,R_I}^{\beta,\gamma}(t) := \left(t, \mathbf{y}(t), \mathbf{y}'(t), B_{P_D}^\beta[\tau \mapsto \mathbf{y}(\tau)](t), K_{R_I}^\gamma[\tau \mapsto \mathbf{y}(\tau)](t)\right),$$

where

$$B^{\beta}_{P_D} := \left(B^{\beta_1}_{P_1}, \ldots, B^{\beta_n}_{P_n}\right), \quad K^{\gamma}_{R_I} := \left(K^{\gamma_1}_{R_1}, \ldots, K^{\gamma_m}_{R_m}\right).$$

The operator K^{α}_P has kernel $k_\alpha(x,t)$ and, for $i=1,\ldots,n$ and $k=1,\ldots,m$, operators $B^{\beta_i}_{P_i}$ and $K^{\gamma_k}_{R_k}$ have kernels $h_{1-\beta_i}(t,\tau)$ and $h_{\gamma_k}(t,\tau)$, respectively. In the sequel we assume that:

(H1) the Lagrangian $F \in C^1\left([a,b]\times\mathbb{R}^{N\times(n+m+2)};\mathbb{R}\right)$;

(H2) functions $D\left[t\mapsto \partial_{N+j}F\left\{\mathbf{y}\right\}^{\beta,\gamma}_{P_D,R_I}(t)k_\alpha(b,t)\right]$, $A^{\beta_i}_{P_i^*}\left[\tau\mapsto k_\alpha(b,\tau)\partial_{(i+1)N+j}F\left\{\mathbf{y}\right\}^{\beta,\gamma}_{P_D,R_I}(\tau)\right]$, $K^{\gamma_k}_{R_k^*}\left[\tau\mapsto k_\alpha(b,\tau)\partial_{(n+1+k)N+j}F\left\{\mathbf{y}\right\}^{\beta,\gamma}_{P_D,R_I}(\tau)\right]$ and $t\mapsto k_\alpha(b,t)\partial_j F\left\{\mathbf{y}\right\}^{\beta,\gamma}_{P_D,R_I}(t)$ are continuous on (a,b), $j=2,\ldots,N+1$, $i=1,\ldots,n$, $k=1,\ldots,m$;

(H3) functions $t\mapsto \partial_{N+j}F\left\{\mathbf{y}\right\}^{\beta,\gamma}_{P_D,R_I}(t)k_\alpha(b,t)$ and $K^{1-\beta_i}_{P_i^*}\left[\tau\mapsto k_\alpha(b,\tau)\partial_{(i+1)N+j}F\left\{\mathbf{y}\right\}^{\beta,\gamma}_{P_D,R_I}(\tau)\right]$ $\in AC([a,b])$, $j=2,\ldots,N+1$, $i=1,\ldots,n$;

(H4) for $i=1,\ldots,n$, $k=1,\ldots,m$, the kernels $k_\alpha(x,t)$, $h_{1-\beta_i}(t,\tau)$ and $h_{\gamma_k}(t,\tau)$ are such that we are able to use Theorems 11, 12, 13 and/or 14.

Definition 15. *A function $\mathbf{y}\in C^1\left([a,b];\mathbb{R}^N\right)$ is said to be admissible for the fractional variational problem (3)–(4) if functions $B^{\beta_i}_{P_i}[\mathbf{y}]$ and $K^{\gamma_k}_{R_k}[\mathbf{y}]$, $i=1,\ldots,n$, $k=1,\ldots,m$ exist and are continuous on the interval $[a,b]$, and $\mathbf{y}$ satisfies the given boundary conditions (4).*

Theorem 16. *If $\mathbf{y}$ is a solution to problem (3)–(4), then $\mathbf{y}$ satisfies the system of generalized Euler–Lagrange equations*

$$\begin{aligned}&k_\alpha(b,t)\partial_j F\left\{\mathbf{y}\right\}^{\beta,\gamma}_{P_D,R_I}(t) - \sum_{i=1}^{n} A^{\beta_i}_{P_i^*}\left[\tau\mapsto k_\alpha(b,\tau)\partial_{(i+1)N+j}F\left\{\mathbf{y}\right\}^{\beta,\gamma}_{P_D,R_I}(\tau)\right](t)\\ &+\sum_{k=1}^{m} K^{\gamma_k}_{R_k^*}\left[\tau\mapsto k_\alpha(b,\tau)\partial_{(n+1+k)N+j}F\left\{\mathbf{y}\right\}^{\beta,\gamma}_{P_D,R_I}(\tau)\right](t) - \frac{d}{dt}\left(\partial_{N+j}F\left\{\mathbf{y}\right\}^{\beta,\gamma}_{P_D,R_I}(t)k_\alpha(b,t)\right) = 0\end{aligned} \tag{5}$$

for all $t\in(a,b)$, $j=2,\ldots,N+1$.

Proof. The proof is analogous to that of [30, Theorem 4.2]. □

4 Generalized free-boundary variational problem

Assume now that in problem (3)–(4) the boundary conditions (4) are substituted by

$$\mathbf{y}(a) \text{ is free and } \mathbf{y}(b) = \mathbf{y}_b. \tag{6}$$

Theorem 17. *If $\mathbf{y}$ is a solution to the problem of extremizing functional (3) with (6) as the boundary conditions, then $\mathbf{y}$ satisfies the system of Euler–Lagrange equations (5). Moreover, the extra system of natural boundary conditions*

$$\partial_{N+j}F\left\{\mathbf{y}\right\}^{\beta,\gamma}_{P_D,R_I}(a)k_\alpha(b,a) + \sum_{i=1}^{n} K^{1-\beta_i}_{P_i^*}\left[\tau\mapsto \partial_{(i+1)N+j}F\left\{\mathbf{y}\right\}^{\beta,\gamma}_{P_D,R_I}(\tau)k_\alpha(b,\tau)\right](a) = 0, \tag{7}$$

$j=2,\ldots,N+1$, holds.

Proof. The proof is analogous to that of [30, Theorem 5.1]. □

5 Generalized isoperimetric problem

Let $\xi \in \mathbb{R}$. Among all functions $\mathbf{y} : [a,b] \to \mathbb{R}^N$ satisfying the boundary conditions

$$\mathbf{y}(a) = \mathbf{y}_a, \quad \mathbf{y}(b) = \mathbf{y}_b, \tag{8}$$

and an isoperimetric constraint of the form

$$\mathcal{I}(\mathbf{y}) = K_P^\alpha \left[G \{\mathbf{y}\}_{P_D, R_I}^{\beta,\gamma} \right](b) = \xi, \tag{9}$$

we look for those that extremize (i.e., minimize or maximize) the functional

$$\mathcal{J}(\mathbf{y}) = K_P^\alpha \left[F \{\mathbf{y}\}_{P_D, R_I}^{\beta,\gamma} \right](b). \tag{10}$$

For $i = 1, \ldots, n$, $k = 1, \ldots, m$ operators K_P^α, $B_{P_i}^{\beta_i}$ and $K_{R_k}^{\gamma_k}$, as well as function F, are the same as in problem (3)–(4). Moreover, we assume that functional (9) satisfies hypotheses (H1)–(H4).

Definition 18. *A function $\mathbf{y} : [a,b] \to \mathbb{R}^N$ is said to be* admissible *for problem* (8)–(10) *if functions $B_{P_i}^{\beta_i}[\mathbf{y}]$ and $K_{R_k}^{\gamma_k}[\mathbf{y}]$, $i = 1, \ldots, n$, $k = 1, \ldots, m$, exist and are continuous on $[a,b]$, and $\mathbf{y}$ satisfies the given boundary conditions* (8) *and the isoperimetric constraint* (9)*.*

Definition 19. *An admissible function $\mathbf{y} \in C^1\left([a,b], \mathbb{R}^N\right)$ is said to be an* extremal *for $\mathcal{I}$ if it satisfies the system of Euler–Lagrange equations* (5) *associated with functional in* (9)*.*

Theorem 20. *If $\mathbf{y}$ is a solution to the isoperimetric problem* (8)–(10) *and is not an extremal for $\mathcal{I}$, then there exists a real constant λ such that*

$$\begin{aligned} &k_\alpha(b,t)\partial_j H \{\mathbf{y}\}_{P_D,R_I}^{\beta,\gamma}(t) + \sum_{k=1}^{m} K_{R_k^*}^{\gamma_k}\left[\tau \mapsto k_\alpha(b,\tau)\partial_{(n+1+k)N+j} H \{\mathbf{y}\}_{P_D,R_I}^{\beta,\gamma}(\tau)\right](t) \\ &- \sum_{i=1}^{n} A_{P_i^*}^{\beta_i}\left[\tau \mapsto k_\alpha(b,\tau)\partial_{(i+1)N+j} H \{\mathbf{y}\}_{P_D,R_I}^{\beta,\gamma}(\tau)\right](t) - \frac{d}{dt}\left(\partial_{j+N} H \{\mathbf{y}\}_{P_D,R_I}^{\beta,\gamma}(t) k_\alpha(b,t)\right) = 0 \end{aligned}$$

for all $t \in (a,b)$, $j = 2, \ldots, N+1$, where $H(t,y,u,v,w) = F(t,y,u,v,w) - \lambda G(t,y,u,v,w)$, $P_i^* =< a,t,b,q_i,p_i >$, and $R_k^* =< a,t,b,s_k,r_k >$.

Proof. The proof is analogous to that of [30, Theorem 6.3]. □

6 Generalized fractional Noether's theorem

Emmy Noether's theorem on extremal functionals, establishing that certain symmetries imply conservation laws (constants of motion), has been called "the most important theorem in physics since the Pythagorean theorem". For a recent account of Noether's theorem and possible applications in physics, from many different points of view, we refer the reader to [28]. Formulations in the more general context of optimal control can be found in [15,40]. Conservation laws appear naturally in closed systems. In presence of non-conservative or dissipative forces, the constants of motion are broken and Noether's classical theorem ceases to be valid. It is still possible, however, to obtain Noether type theorems that cover both conservative and non-conservative cases. Roughly speaking, one can prove that Noether's conservation laws are still valid if a new term, involving the non-conservative forces, is added to the standard constants of motion [10]. The first Noether theorem for the fractional calculus of variations was obtained in 2007 [11]. Since then, the subject attracted a lot of attention. The state of the art is given in the book [24]. Here we obtain a Noether's theorem for generalized fractional variational problems.

Definition 21. *We say that the functional* (3) *is invariant under an ε-parameter group of infinitesimal transformations*

$$\hat{\mathbf{y}}(t) = \mathbf{y}(t) + \varepsilon \boldsymbol{\xi}(t, \mathbf{y}(t)) + o(\varepsilon) \tag{11}$$

if for any subinterval $[t_a, t_b] \subseteq [a, b]$ one has

$$K_{\bar{P}}^{\alpha} \left[t \mapsto F \left\{ \mathbf{y} \right\}_{P_D, R_I}^{\beta,\gamma} (t) \right] (t_b) = K_{\bar{P}}^{\alpha} \left[t \mapsto F \left\{ \hat{\mathbf{y}} \right\}_{P_D, R_I}^{\beta,\gamma} (t) \right] (t_b), \tag{12}$$

where $\bar{P} = \langle t_a, t_b, t_b, 1, 0 \rangle$.

Theorem 22. *If functional* (3) *is invariant under an ε-parameter group of infinitesimal transformations, then*

$$\begin{aligned} \sum_{j=2}^{N+1} \Bigg(& \partial_j F \left\{ \mathbf{y} \right\}_{P_D, R_I}^{\beta,\gamma} (t) \cdot \xi_{j-1}(t, \mathbf{y}(t)) + \partial_{N+j} F \left\{ \mathbf{y} \right\}_{P_D, R_I}^{\beta,\gamma} (t) \cdot \frac{d}{dt} \xi_{j-1}(t, \mathbf{y}(t)) \\ & + \sum_{i=1}^{n} \partial_{(i+1)N+j} F \left\{ \mathbf{y} \right\}_{P_D, R_I}^{\beta,\gamma} (t) \cdot B_{P_i}^{\beta_i} [\tau \mapsto \xi_{j-1}(\tau, \mathbf{y}(\tau))](t) \\ & + \sum_{k=1}^{m} \partial_{(n+1+k)N+j} F \left\{ \mathbf{y} \right\}_{P_D, R_I}^{\beta,\gamma} (t) \cdot K_{R_i}^{\gamma_i} [\tau \mapsto \xi_{j-1}(\tau, \mathbf{y}(\tau))](t) \Bigg) = 0. \end{aligned} \tag{13}$$

Proof. Since, by hypothesis, condition (12) is satisfied for any subinterval $[t_a, t_b] \subseteq [a, b]$, we have

$$F \left\{ \mathbf{y} \right\}_{P_D, R_I}^{\beta,\gamma} (t) = F \left\{ \hat{\mathbf{y}} \right\}_{P_D, R_I}^{\beta,\gamma} (t). \tag{14}$$

Differentiating (14) with respect to ε, then putting $\varepsilon = 0$, and applying definitions and properties of generalized fractional operators, we obtain (13). □

In order to state the Noether theorem in a compact form, we introduce the following operators:

$$\mathbf{D}_P^{\alpha}[f, g](t) := \frac{1}{k_\alpha(b, t)} f(t) \cdot A_{P^*}^{\alpha}[g](t) + g(t) \cdot B_P^{\alpha}[f](t), \tag{15}$$

$$\mathbf{I}_P^{\alpha}[f, g](t) := \frac{-1}{k_\alpha(b, t)} f(t) \cdot K_{P^*}^{\alpha}[g](t) + g(t) \cdot K_P^{\alpha}[f](t), \tag{16}$$

where P^* denotes the dual p-set of P, that is, if $P = \langle a, t, b, p, q \rangle$, then $P^* = \langle a, t, b, q, p \rangle$.

Theorem 23 (Generalized fractional Noether's theorem)**.** *If functional* (3) *is invariant under an ε-parameter group of infinitesimal transformations* (11)*, then*

$$\begin{aligned} \sum_{j=2}^{N+1} \Bigg(& \sum_{i=1}^{n} \mathbf{D}_{P_i}^{\beta_i} \left[\tau \mapsto \xi_{j-1}(\tau, \mathbf{y}(\tau)), \tau \mapsto k_\alpha(b, \tau) \partial_{(i+1)N+j} F \left\{ \mathbf{y} \right\}_{P_D, R_I}^{\beta,\gamma} (\tau) \right] (t) \\ & + \sum_{k=1}^{m} \mathbf{I}_{R_k}^{\gamma_k} \left[\tau \mapsto \xi_{j-1}(\tau, \mathbf{y}(\tau)), \tau \mapsto k_\alpha(b, \tau) \partial_{(n+1+k)N+j} F \left\{ \mathbf{y} \right\}_{P_D, R_I}^{\beta,\gamma} (\tau) \right] (t) \\ & + \frac{d}{dt} \left(\xi_{j-1} \left(t, \mathbf{y}(t) \right) \cdot \partial_{N+j} F \left\{ \mathbf{y} \right\}_{P_D, R_I}^{\beta,\gamma} (t) \right) + \xi_{j-1}(t, \mathbf{y}(t)) \cdot \partial_{N+j} F \left\{ \mathbf{y} \right\}_{P_D, R_I}^{\beta,\gamma} (t) \cdot \frac{1}{k_\alpha(b, t)} \frac{d}{dt} k_\alpha(b, t) \Bigg) = 0 \end{aligned} \tag{17}$$

for any generalized fractional extremal $\mathbf{y}$ of $\mathcal{J}$ and for all $t \in (a, b)$.

Proof. By Theorem 16 we have

$$k_\alpha(b,t)\partial_j F\{\mathbf{y}\}^{\beta,\gamma}_{P_D,R_I}(t)=\sum_{i=1}^{n}A^{\beta_i}_{P_i^*}\left[\tau\mapsto k_\alpha(b,\tau)\partial_{(i+1)N+j}F\{\mathbf{y}\}^{\beta,\gamma}_{P_D,R_I}(\tau)\right](t)$$
$$-\sum_{k=1}^{m}K^{\gamma_k}_{R_i^*}\left[\tau\mapsto k_\alpha(b,\tau)\partial_{(n+1+k)N+j}F\{\mathbf{y}\}^{\beta,\gamma}_{P_D,R_I}(\tau)\right](t)+\frac{d}{dt}\left(\partial_{N+j}F\{\mathbf{y}\}^{\beta,\gamma}_{P_D,R_I}(t)k_\alpha(b,t)\right) \tag{18}$$

for all $t\in(a,b)$, $j=2,\ldots,N+1$. Substituting (18) into (12), we obtain

$$\sum_{j=2}^{N+1}\left[\frac{1}{k_\alpha(b,t)}\cdot\xi_{j-1}(t,\mathbf{y}(t))\left(\sum_{i=1}^{n}A^{\beta_i}_{P_i^*}\left[\tau\mapsto k_\alpha(b,\tau)\partial_{(i+1)N+j}F\{\mathbf{y}\}^{\beta,\gamma}_{P_D,R_I}(\tau)\right](t)\right.\right.$$
$$\left.-\sum_{k=1}^{m}K^{\gamma_k}_{R_i^*}\left[\tau\mapsto k_\alpha(b,\tau)\partial_{(n+1+k)N+j}F\{\mathbf{y}\}^{\beta,\gamma}_{P_D,R_I}(\tau)\right](t)+\frac{d}{dt}\left(\partial_{N+j}F\{\mathbf{y}\}^{\beta,\gamma}_{P_D,R_I}(t)k_\alpha(b,t)\right)\right)$$
$$+\partial_{N+j}F\{\mathbf{y}\}^{\beta,\gamma}_{P_D,R_I}(t)\cdot\frac{d}{dt}\xi_{j-1}(t,\mathbf{y}(t))+\sum_{i=1}^{n}\partial_{(i+1)N+j}F\{\mathbf{y}\}^{\beta,\gamma}_{P_D,R_I}(t)\cdot B^{\beta_i}_{P_i}[\tau\mapsto\xi_{j-1}(\tau,\mathbf{y}(\tau))](t)$$
$$\left.+\sum_{k=1}^{m}\partial_{(n+1+k)N+j}F\{\mathbf{y}\}^{\beta,\gamma}_{P_D,R_I}(t)\cdot K^{\gamma_i}_{R_i}[\tau\mapsto\xi_{j-1}(\tau,\mathbf{y}(\tau))](t)\right]=0.$$

Finally, we arrive to (17) by (15) and (16). □

Example 24. *Let $P=\langle a,t,b,p,q\rangle$. Consider the following problem:*

$$\mathcal{J}[y]=\int_a^b F\left(t,B^\alpha_P[y](t)\right)dt\longrightarrow\min \qquad y(a)=y_a\,,\quad y(b)=y_b, \tag{19}$$

and transformations

$$\hat{y}(t)=y(t)+\varepsilon c+o(\varepsilon), \tag{20}$$

where c is a constant. For any $[t_a,t_b]\subseteq[a,b]$ we have

$$\int_{t_a}^{t_b}F\left(t,B^\alpha_P[y](t)\right)dt=\int_{t_a}^{t_b}F\left(t,B^\alpha_P[\hat{y}](t)\right)dt.$$

Therefore, $\mathcal{J}[y]$ is invariant under (20) and Theorem 23 asserts that

$$A^\alpha_{P^*}[\tau\to\partial_2F\left(\tau,B^\alpha_P[y](\tau)\right)](t)=0 \tag{21}$$

along any generalized fractional extremal y. Notice that equation (21) can be written in the form

$$\frac{d}{dt}\left(K^\alpha_{P^*}[\tau\to\partial_2F\left(\tau,B^\alpha_P[y](\tau)\right)](t)\right)=0.$$

In analogy with the classical approach, quantity $K^\alpha_{P^*}[\tau\to\partial_2F\left(\tau,B^\alpha_P[y](\tau)\right)](t)$ is called a generalized fractional constant of motion.

7 Applications to Physics

If the functional (3) does not depend on B-ops and K-ops, then Theorem 16 gives the following result: if $\mathbf{y}$ is a solution to the problem of extremizing

$$\mathcal{J}(\mathbf{y})=\int_a^b L\left(t,\mathbf{y}(t),\mathbf{y}'(t)\right)k_\alpha(b,t)dt \tag{22}$$

subject to $\mathbf{y}(a)=\mathbf{y}_a$ and $\mathbf{y}(b)=\mathbf{y}_b$, where $\alpha\in(0,1)$, then

$$\partial_j L\left(t,\mathbf{y}(t),\mathbf{y}'(t)\right)-\frac{d}{dt}\partial_{N+j}L\left(t,\mathbf{y}(t),\mathbf{y}'(t)\right)=\frac{1}{k_\alpha(b,t)}\cdot\frac{d}{dt}k_\alpha(b,t)\partial_{N+j}L\left(t,\mathbf{y}(t),\mathbf{y}'(t)\right),\qquad(23)$$

$j=2,\ldots,N+1$. In addition, if we assume that functional (22) is invariant under transformations (11), then Noether's theorem yields that

$$\sum_{j=2}^{N+1}\left(\frac{d}{dt}\left(\xi_{j-1}(t,\mathbf{y}(t))\cdot\partial_{N+j}L(t,\mathbf{y}(t),\mathbf{y}'(t))\right)+\xi_{j-1}(t,\mathbf{y}(t))\cdot\partial_{N+j}L(t,\mathbf{y}(t),\mathbf{y}'(t))\cdot\frac{1}{k_\alpha(b,t)}\frac{d}{dt}k_\alpha(b,t)\right)=0,$$

along any extremal of (22). Let us consider kernel $k_\alpha(b,t)=\mathrm{e}^{\alpha(b-t)}$ and the Lagrangian for a three dimensional system:

$$L\left(\mathbf{y},\dot{\mathbf{y}}\right)=\frac{1}{2}m\left(\dot{y_1}^2+\dot{y_2}^2+\dot{y_3}^2\right)-V(\mathbf{y}),$$

where $V(\mathbf{y})$ is the potential energy and m stands for the mass. Observe that an explicitly time dependent integrand $\tilde{L}=\mathrm{e}^{\alpha(b-t)}L$ of functional (22) is known in the literature as the Bateman–Caldirola–Kanai (BCK) Lagrangian of a quantum dissipative system [13,25]. But in our case the Lagrangian of the system is L and not $\mathrm{e}^{\alpha(b-t)}L$. The Euler–Lagrange equations (23) give the following system of second order ordinary differential equations:

$$\begin{cases}\ddot{y_1}(t)-\alpha\dot{y_1}(t)=-\frac{1}{m}\partial_1V(\mathbf{y}(t))\\ \ddot{y_2}(t)-\alpha\dot{y_2}(t)=-\frac{1}{m}\partial_2V(\mathbf{y}(t))\\ \ddot{y_3}(t)-\alpha\dot{y_3}(t)=-\frac{1}{m}\partial_3V(\mathbf{y}(t)).\end{cases}$$

If $\gamma:=-\alpha$, then

$$\ddot{y_i}+\gamma\dot{y_i}+\frac{1}{m}\frac{\partial V}{\partial y_i}=0,\qquad(24)$$

$i=1,2,3$, which are equations for the damped motion of a three-dimensional particle under the action of a force $\left[-\frac{\partial V}{\partial y_1},-\frac{\partial V}{\partial y_2},-\frac{\partial V}{\partial y_3}\right]$ (see, e.g., [17]). Choosing $V:=k\frac{y_1^2+y_2^2+y_3^2}{2}$, we can transform (24) into equations for a damped simple harmonic oscillator:

$$\ddot{y_i}(t)+\gamma\dot{y_i}(t)+\omega^2y_i(t)=0,$$

$i=1,2,3$, with $\omega^2=\frac{k}{m}$. Now, let us consider the following Lagrangian:

$$L\left(\mathbf{y},\dot{\mathbf{y}}\right)=\frac{1}{2}m\left(\dot{y_1}^2+\dot{y_2}^2+\dot{y_3}^2\right)-mgy_3^2.\qquad(25)$$

We see at once that the Lagrangian (25) is invariant under the transformation

$$\hat{y}_1=y_1+\varepsilon,\ \hat{y}_2=y_2,\ \hat{y}_3=y_3.$$

In this case Noether's theorem gives

$$\frac{d}{dt}(m\dot{y_1})=\alpha m\dot{y_1}.\qquad(26)$$

If $\alpha=0$, then there is no friction and (26) yields the classical conservation of linear momentum $p_1=m\dot{y_1}=const$. Observe that the generalized momentum conjugate to y_i is $p_i=\frac{\partial L}{\partial\dot{y_i}}=m\dot{y_i}$, $i=1,2,3$. This is not the case for the the BCK Lagrangian [13], where the canonical momentum for y_i is $\tilde{p}_i=\mathrm{e}^{\alpha(b-t)}m\dot{y_i}$, $i=1,2,3$, that is different from the kinetic momentum. Now, let us suppose that L is variationally invariant under the transformation

$$\hat{y}_1=y_1\cos\varepsilon+y_2\sin\varepsilon,\ \hat{y}_2=-y_1\sin\varepsilon+y_2\cos\varepsilon,\ \hat{y}_3=y_3.$$

Then $\xi_1=y_2$, $\xi_2=-y_1$ and $\xi_3=0$. For this case Noether's theorem yields

$$\frac{d}{dt}(m\dot{y_1}y_2-my_1\dot{y_2})-\alpha m(\dot{y_1}y_2-y_1\dot{y_2})=0.\qquad(27)$$

Note that for $\alpha=0$ relation (27) gives the standard conservation law $p_1y_2-p_2y_1=const$ yielded by the classical Noether's theorem [41, Section 9.3].

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References

[1] O. P. Agrawal, Generalized variational problems and Euler-Lagrange equations, Comput. Math. Appl. **59** (2010), no. 5, 1852–1864.

[2] R. Almeida, A. B. Malinowska and D. F. M. Torres, A fractional calculus of variations for multiple integrals with application to vibrating string, J. Math. Phys. **51** (2010), no. 3, 033503, 12 pp. `arXiv:1001.2722`

[3] R. Almeida, A. B. Malinowska and D. F. M. Torres, Fractional Euler-Lagrange differential equations via Caputo derivatives, In: Fractional Dynamics and Control, Springer New York, 2012, Part 2, 109–118. `arXiv:1109.0658`

[4] R. Almeida and D. F. M. Torres, Calculus of variations with fractional derivatives and fractional integrals, Appl. Math. Lett. **22** (2009), no. 12, 1816–1820. `arXiv:0907.1024`

[5] N. R. O. Bastos, R. A. C. Ferreira and D. F. M. Torres, Necessary optimality conditions for fractional difference problems of the calculus of variations, Discrete Contin. Dyn. Syst. **29** (2011), no. 2, 417–437. `arXiv:1007.0594`

[6] N. R. O. Bastos, R. A. C. Ferreira and D. F. M. Torres, Discrete-time fractional variational problems, Signal Process. **91** (2011), no. 3, 513–524. `arXiv:1005.0252`

[7] J. L. Cieśliński and T. Nikiciuk, A direct approach to the construction of standard and non-standard Lagrangians for dissipative-like dynamical systems with variable coefficients, J. Phys. A **43** (2010), no. 17, 175205, 15 pp.

[8] M. Crampin, T. Mestdag and W. Sarlet, On the generalized Helmholtz conditions for Lagrangian systems with dissipative forces, ZAMM Z. Angew. Math. Mech. **90** (2010), no. 6, 502–508.

[9] J. Cresson, Fractional embedding of differential operators and Lagrangian systems, J. Math. Phys. **48** (2007), no. 3, 033504, 34 pp. `arXiv:math/0605752`

[10] G. S. F. Frederico and D. F. M. Torres, Nonconservative Noether's theorem in optimal control, Int. J. Tomogr. Stat. **5** (2007), no. W07, 109–114. `arXiv:math/0512468`

[11] G. S. F. Frederico and D. F. M. Torres, A formulation of Noether's theorem for fractional problems of the calculus of variations, J. Math. Anal. Appl. **334** (2007), no. 2, 834–846. `arXiv:math/0701187`

[12] G. S. F. Frederico and D. F. M. Torres, Fractional optimal control in the sense of Caputo and the fractional Noether's theorem, Int. Math. Forum **3** (2008), no. 10, 479–493. `arXiv:0712.1844`

[13] S. Ghosh, A. Choudhuri and B. Talukdar, On the quantization of damped harmonic oscillator, Acta Phys. Polon. B **40** (2009), no. 1, 49–57.

[14] M. Giaquinta and S. Hildebrandt, *Calculus of variations. I*, Springer, Berlin, 1996.

[15] P. D. F. Gouveia, D. F. M. Torres and E. A. M. Rocha, Symbolic computation of variational symmetries in optimal control, Control Cybernet. **35** (2006), no. 4, 831–849. `arXiv:math/0604072`

[16] A. Ya. Helemskii, *Lectures and Excercises on Functional Analysis*, American Mathematical Society, 2006.

[17] L. Herrera, L. Núñez, A. Patiño and H. Rago, A variational principle and the classical and quantum mechanics of the damped harmonic oscillator, Am. J. Phys. **54** (1986), no. 3, 273–277.

[18] U. N. Katugampola, New approach to a generalized fractional integral, Appl. Math. Comput. **218** (2011), no. 3, 860–865.

[19] A. A. Kilbas and M. Saigo, Generalized Mittag-Leffler function and generalized fractional calculus operators, Integral Transform. Spec. Func. **15** (2004), no. 1, 31–49.

[20] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[21] M. Klimek, *On solutions of linear fractional differential equations of a variational type*, The Publishing Office of Czestochowa University of Technology, Czestochowa, 2009.

[22] D. H. Kobe, G. Reali and S. Sieniutycz, Lagrangians for dissipative systems, Am. J. Phys. 54 (1986), 997–999.

[23] C. Lanezos, *The variational principles of mechanics*, 4th edition, Dover, New York, 1970.

[24] A. B. Malinowska and D. F. M. Torres, *Introduction to the fractional calculus of variations*, Imp. Coll. Press, London, 2012.

[25] V. J. Menon, N. Chanana and Y. Singh, A Fresh Look at the BCK Frictional Lagrangian, Prog. Theor. Phys. 98 (1997), no. 2, 321–329.

[26] D. Mozyrska and D. F. M. Torres, Modified optimal energy and initial memory of fractional continuous-time linear systems, Signal Process. **91** (2011), no. 3, 379–385. `arXiv:1007.3946`

[27] Z. E. Musielak, Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients, J. Phys. A **41** (2008), no. 5, 055205, 17 pp.

[28] D. E. Neuenschwander, Emmy Noether's wonderful theorem, Johns Hopkins University Press, Baltimore, MD, 2011.

[29] T. Odzijewicz, A. B. Malinowska and D. F. M. Torres, Fractional variational calculus with classical and combined Caputo derivatives, Nonlinear Anal. **75** (2012), no. 3, 1507–1515. `arXiv:1101.2932`

[30] T. Odzijewicz, A. B. Malinowska and D. F. M. Torres, Fractional calculus of variations in terms of a generalized fractional integral with applications to Physics, Abstr. Appl. Anal. **2012** (2012), Art. ID 871912, 24 pp. `arXiv:1203.1961`

[31] T. Odzijewicz, A. B. Malinowska and D. F. M. Torres, Generalized fractional calculus with applications to the calculus of variations, Comput. Math. Appl. **64** (2012), no. 10, 3351–3366. `arXiv:1201.5747`

[32] T. Odzijewicz, A. B. Malinowska and D. F. M. Torres, Fractional variational calculus of variable order, Advances in Harmonic Analysis and Operator Theory, The Stefan Samko Anniversary Volume (Eds: A. Almeida, L. Castro, F.-O. Speck), Operator Theory: Advances and Applications, Vol. 229 (2013), 291–301. `arXiv:1110.4141`

[33] T. Odzijewicz and D. F. M. Torres, Fractional calculus of variations for double integrals, Balkan J. Geom. Appl. **16** (2011), no. 2, 102–113. `arXiv:1102.1337`

[34] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, CA, 1999.

[35] A. D. Polyanin and A. V. Manzhirov, *Handbook of integral equations*, CRC, Boca Raton, FL, 1998.

[36] S. Pooseh, R. Almeida and D. F. M. Torres, Expansion formulas in terms of integer-order derivatives for the Hadamard fractional integral and derivative, Numer. Funct. Anal. Optim. **33** (2012), no. 3, 301–319. `arXiv:1112.0693`

[37] R. G. Pradeep, V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, Nonstandard conserved Hamiltonian structures in dissipative/damped systems: nonlinear generalizations of damped harmonic oscillator, J. Math. Phys. **50** (2009), no. 5, 052901, 15 pp.

[38] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, Phys. Rev. E (3) **53** (1996), no. 2, 1890–1899.

[39] F. Riewe, Mechanics with fractional derivatives, Phys. Rev. E (3) **55** (1997), no. 3, part B, 3581–3592.

[40] D. F. M. Torres, On the Noether theorem for optimal control, Eur. J. Control **8** (2002), no. 1, 56–63.

[41] B. van Brunt, *The calculus of variations*, Universitext, Springer, New York, 2004.